

## 2. MATHS 2: CLOSING IN

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In this chapter we continue to talk about some basic maths. The maths of this chapter are more closely related to physics and thus serve both as a bridge to our consideration of the physics itself and a closure with that physics.

**Remark 1** *One of the complaints that is often voiced is that because maths are hard and complicated, then we should study them no more than is absolutely necessary. This argument, while apparently logical, is actually pure rubbish. In actuality, the more maths that are learned, the simpler and less complicated they become, both to learn and to use. The (classical) Greeks never got beyond the idea of specific values of numbers and thus never got to variables or invented algebra, much less calculus or any of the more advanced maths. As a result their science never got much beyond philosophy and ad hoc engineering. Problems that they could not solve at all with only specific values are trivial with algebra (such as musical scales!), while problems that are hideously difficult with algebra (like the area under a graphed curve,) are trivial with calculus.*

Thus, we study maths not because they are hard, but because knowing them makes things easy.

### 2.1 Measurement: Coordinate Systems and Units

William Thomson, the first Lord Kelvin, an accomplished English physicist, while lecturing to the Institution of Civil Engineers on the Third of May, 1883, said

"When you can measure what you are speaking about and express it in numbers, you know something about it."

One of the things that sets science apart from many other human activities is measurement. While there is a role in science for description, otherwise we should not be writing this book, description without some form of measurement is fundamentally a sterile dead end.

This is not to say that one may not be confronted with a dead end, either permanent or temporary, after measurement, but rather that without measurement there is nothing but dead end of the permanent sort.

However, to measure things we must have some system of measurement. There are two necessary components of a measurement system:<sup>1</sup>

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<sup>1</sup>There may arise a requirement for more than these two, but all measurement systems require at least these two.

- an ordered framework for the measurement; and
- an ordered framework of the measurement.

The first of these provides a context for the measurement while the second provides a magnitude of the measurement. We shall discuss these sequentially.

### 2.1.1 Coordinate Systems

The ordered framework for a measurement is provided by a **coordinate system**. The simplest and most common coordinate systems, which are what we shall primarily consider here, are spatial coordinate systems. That is, they are coordinate systems that allow us to measure space. These are the simplest because they are directly visualizable by humans.

### 2.1.2 Spatial Coordinate Systems

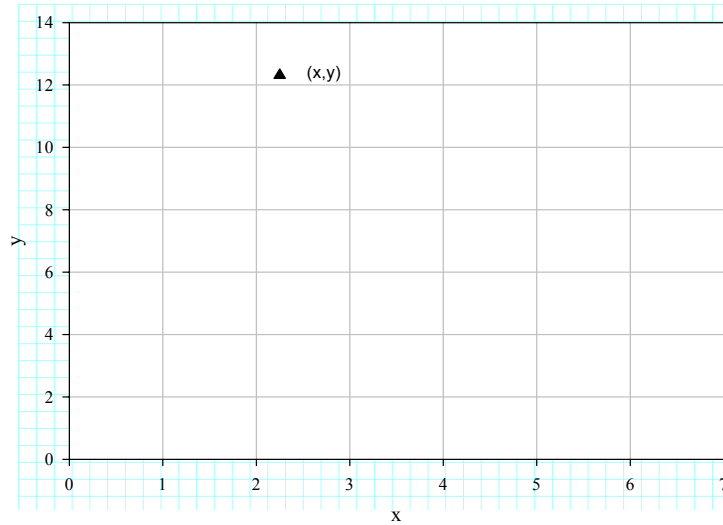
The simplest spatial coordinate system is represented by a line with equally spaced, finite increments imposed on it. We are not only quite familiar with this, but recognize it to be rather limited in usefulness since very few problems are typified by having only one **dimension** of length.

**Remark 2** *Yes, we all measure lengths with things like rulers, or yard or meter sticks, or steel or cloth tapes. We measure the length of a piece of wood or cloth, the height of a person, or the dimensions of a room. We may get away with only measuring the single dimension because all of these things have fixed dimensions. In most cases, they are solids although we shall not consider in detail the characteristics of solids for a few chapters yet.*

The next more complicated coordinate systems are the planar, which we have already discussed in the previous chapter. The graph is a coordinate system that provides a means of visualizing a set of point specific values of two variables. If we plot or graph enough points, or connect the points with lines, then the graph becomes that of a curve. If we now invert this idea to consider the axes of the graph to provide a means of specifying the particular values of the positions of points on the graph, then we have an example of a coordinate system and its use.

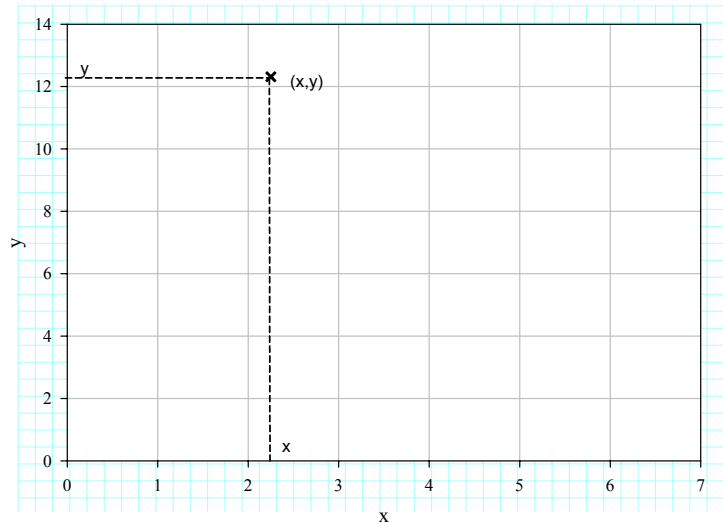
### Two-Dimensional (2-D) Spatial Coordinate Systems

If we now visualize a graph - a pair of incremented axes - with one point on it, as shown in the figure,



Example of a point in a (graph) coordinate system

then we may envision two obvious ways of specifying where that point is in the coordinate system of the axes of the graph. The first of these is by the projection of the point onto each axis. This specifies the particular values of the point in the coordinate system, as shown in this modified figure.



Example of a point in a (graph) coordinate system with projection of the point onto the two coordinate axes

It may be recalled from the previous chapter that we then drew a graph from a set of points  $\{x_i, f(x_i), i = 1..I\}$  (where the curly braces  $\{\}$  indicate a set or collection.) We also wrote a function of  $x$  as  $y = f(x)$ , where we considered  $y$  to be a separate variable. Now, let us assume that  $y$  may be a variable independent of  $x$ , so that we may talk about a point on a graph  $(x, y)$  where  $x$  and  $y$  are not (necessarily) related to each other except by the framework of the coordinate system. Then we may conceive of measuring  $x$  and

$y$  using that coordinate system (usually through some measuring instrument such as a ruler,) so that the position of the point is specified.

Notice here, that we have put some rigor into what we previously said about one dimensional (1-D) measurement. Such measurements generally are only meaningful in some fixed context and that context here is the coordinate system.

If we have a coordinate system where the axes have the same units of measurement, and no axis has any overlap (called **projection**) with any of the other axes other than a point (at the origin, e.g.,) then we say that such a coordinate system is **Cartesian**. In our case the 2-D coordinate system just introduced is Cartesian if the two axes have the same units of measurement.

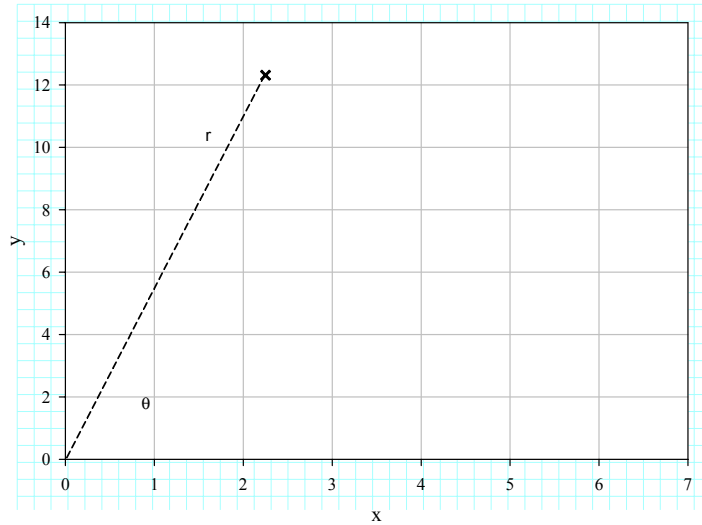
In geometric terms, the two axes are at right angles to each other. We sometimes say that the two axes are **orthogonal** to each other and we tend to think that orthogonality is associated with the right angle. This association is not wrong when we are dealing with purely spatial coordinate systems, but when we deal with coordinates that are not spatial, this association can lead to erroneous interpretation. Indeed, as we proceed, we shall introduce the idea of functional orthogonality. A better way of viewing this is that coordinates that are orthogonal to each other are independent of each other in the context of the coordinate system. They may be linked by the physics, but not by the coordinate system.

Thus, the statement that the  $x$  and  $y$  coordinate axes are orthogonal to each other means that the measurements of specific values of  $x$  and  $y$  do not depend on each other.

We mentioned that there was a second obvious way to measure the position of a point. That way is to measure the distance of the point from the origin, and the angle that the line from the origin to the point makes with the x-axis as shown in the figure.<sup>2</sup>

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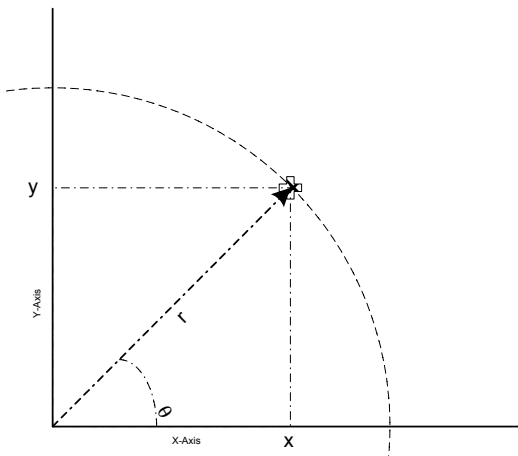
<sup>2</sup>Which axis we measure from is arbitrary, but tradition and convention are that we measure from the x-axis, which is the horizontal in our way of drawing.



Example of a point in a (graph) coordinate system specified by its distance from the origin and angle to the x-axis

Obviously this coordinate system is not Cartesian, since a distance and an angle will not be measured in the same units. The axes do not appear to be orthogonal but we defer addressing this matter till a later section of this chapter. We usually call this coordinate system the **circular** coordinate system as we call the previous one the **rectangular** coordinate system.

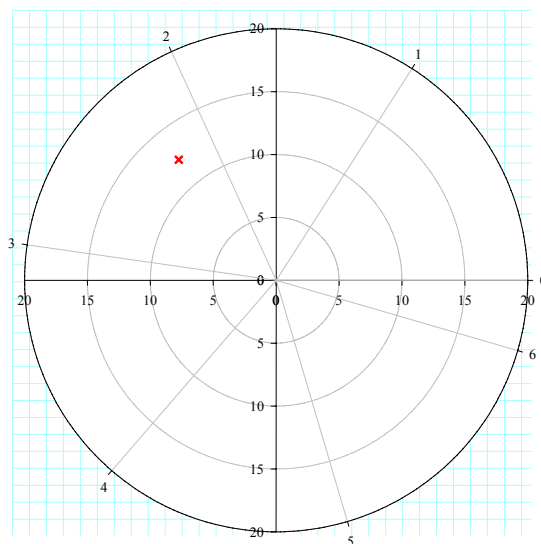
Thus, we may specify the position of a point in 2-D by either its rectangular coordinates  $(x, y)$  or its circular coordinates  $(r, \theta)$ . For convenience we present the two systems imposed on the same graph in this figure.



Combined portrayal of the rectangular and circular coordinate systems.

From this it may be seen that the geometry of the rectangular coordinate system is a rectangular grid (square if Cartesian), while the geometry of the circular coordinate system would be concentric circles of radius  $m \Delta r$  and "pie"

wedges of angle  $n \Delta\theta$ . This is shown in this figure.



Circular coordinate system showing the geometry. Theta goes from zero to two pi radians.

We may also see that the specific coordinate values are related by

$$\begin{aligned}
 r &= \sqrt{x^2 + y^2}, \\
 \theta &= \arctan\left(\frac{y}{x}\right), \\
 x &= r \cos(\theta), \\
 y &= r \sin(\theta).
 \end{aligned}
 \tag{2.1}$$

If we think about this and take the coordinate axes as a given, then the combination of the coordinate system and the point may be viewed as forming a right triangle where the projection of the point onto the x-axis is the base and the projection of the point onto the y-axis is the height. Thus, we may specify the position of the point either as the base and the height of the triangle, or as the hypotenuse and the angle (between the hypotenuse and the base, sometimes called the included angle.)

Further, we note that we need two coordinate values to specify the position of a point in 2-D. This is a general rule. If a space has  $n$  dimensions, then we need  $n$  coordinate values to specify a position in the space.

### 2.1.3 Spatial Units of Measure

As we said previously, the other half of the measurement system is the units of measure. In essence, what are the units the coordinate axes are divided into? For spatial measurements, the units are usually called units of length,

and there are two major systems of length units, the British (Engineering) and the Metric.

The basic units of the British system are the inch, the foot (plural feet), and the yard, which are related by

- 12 inch (in.) = 1 foot (ft.),
- 3 feet = 1 yard (yd.)

There is also a unit called a (land) mile (which is different from a sea or nautical mile,) that is related by

- 5280 ft. = 1 mile (mi.)

It may immediately be seen that the British system of measurement is not very well ordered. This is largely due to its development over time as a practical system of measurement. As such it was not intended as a scientific system of measurement, It is still in use today in the United States, primarily for economic and social reasons

The other system, the metric system, was developed expressly for scientific purposes. It is used throughout the world as a scientific system of measurement and as a general system of measurement outside the United States. The basic length units of the metric system are the millimeter (mm.), the meter (m.), and the kilometer (km.), related by

- 1.0 mm. =  $10^{-3}$  m.,
- 1.0 m. =  $10^{-3}$  km.

The regularity of the system as well as its applicability in measurement is thus obvious. The two systems are related by

- 25.4 mm  $\simeq$  1.0 in.
- 1.0 m.  $\simeq$  39.37 in.

We shall return to our consideration of units of measurement later. For now, we turn back to our consideration of coordinate systems.

#### 2.1.4 Three Dimensional (3-D) Spatial Coordinate Systems

To make the leap from 2-D to 3-D we need only add another axis (call it "z") to the existing x- and y-axes.

Ah, but it isn't quite that easy because there are two ways we can add the axis. If we consider the 2-D rectangular coordinate system, with the (positive)

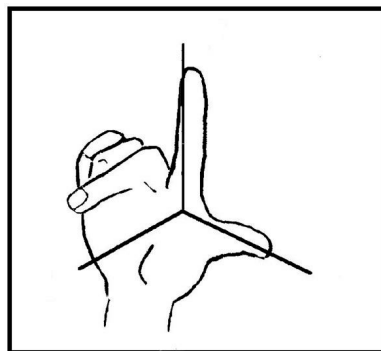
x-axis being horizontal in the page, and the (positive) y-axis being vertical in the page, then do we append the (positive) z-axis, which we want to be perpendicular (orthogonal) to the other two, to point into the page or out of the page?

For a reason that we shall demonstrate by subsequent example, we want the latter. That is, we want the (positive) z-axis to come out of the page. The reason we want this is because this is what is called a **right handed coordinate system** (the other option is left handed.) We want to use right handed coordinate systems for reasons of convention, not because they are better than left handed coordinate systems, but just because of the weight of tradition.

**Remark 3** *When I was first introduced to 3-D coordinate systems it was in a room whose walls were made of cinder blocks and whose floor was covered with squares of linoleum whose edges were quite obvious. Thus, each of the four (floor) corners of the room formed a graduated (but not Cartesian) coordinate system (because the linoleum squares were wider than the cinder blocks were high.)*

Now let us visualize for a moment. Go to a corner of the room. Sit down on the floor so that your legs are together and pointed towards the opposite corner. Sit with your back against the two walls. Now, the boundary line between the wall on your right and the floor is the x-axis, and between the wall on your left and the floor is the y-axis, and the boundary line between the two walls is the z-axis.

Now extend your lower right arm to be parallel to the x-axis line (Its ok for your arm to almost touch the right wall.) Open your hand and turn it at the wrist so the palm is away from the wall. Now curl the bottom two fingers into the palm. Extend the first finger straight ahead and the second finger perpendicular to the first finger, pointing to the left. Now raise the thumb. Your hand should not look like the figure.



Extension of the first two fingers and thumb of the right hand

Envision now that the first finger represents the x-axis, the second finger the y-axis, and the thumb the z-axis.



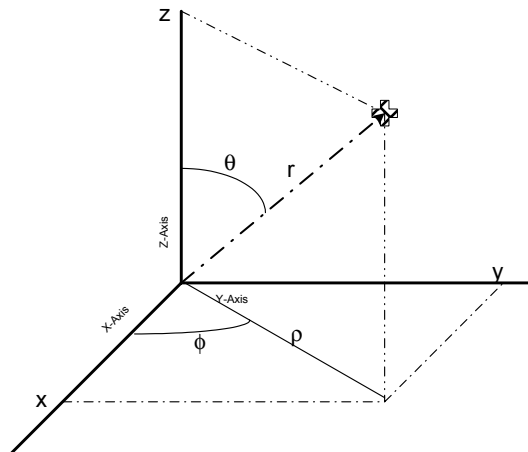
Hence, a right handed coordinate system.

Now straighten the fingers of your hand. Keep the back of the hand aligned to the right wall, and partially curl your four fingers until the tips point out of the wall. Now say the words "x cross y into z." We shall explain the meaning of this shortly.<sup>3</sup>

Consider a point in the 3-D coordinate system. It is a given that to specify its position in space we must measure the projection of this point's position onto each of the three axes. Thus, we may measure the coordinate values  $(x, y, z)$ , at least in a rectangular coordinate system (yes, its still called that even in 3-D.)

As we have noted previously, we may specify the point's position by several forms of these three numbers. We may, for example, again use triangle constructions as we have before. Construct a right triangle with three points: the origin, the point under consideration, and the projection of that point onto the z-axis. We may designate the hypotenuse as  $r$  and the included angle as  $\theta$ , as before, except that now we note that  $\theta$  has range  $[0, \pi]$  where it had range  $[0, 2\pi]$  in the circular coordinate system.

This only defines two numbers however. If we now project the point under consideration into the x-y plane, and define another triangle, this time with angle  $\phi$  (range  $[0, \pi]$ ) with respect to the x-axis, then the combination  $(r, \theta, \phi)$  specifies the position of the point under consideration. Both of these coordinate systems are shown in the figure.



Structure of 3-D rectangular and spherical coordiante systems

This latter coordinate system is called the **spherical** coordinate system because of its geometry, part of which is nested spheres of radius  $m \Delta r$ . The rest of the geometry is somewhat more complicated and we shall discuss this in some depth shortly.

Obviously, the geometry of the rectangular 3-D coordinate system is rectangular boxes, unless the system is Cartesian, when it is cubes.

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<sup>3</sup>For those concerned, this has no religious significance, only mathematical.

We may easily see from this figure that the relationship between the two coordinate systems is straightforward and the demonstration of the following relationships is left for the student. For the spherical coordinate system, we have

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2}, \\ \cos(\theta) &= \frac{z}{r}, \\ \tan(\phi) &= \frac{y}{x}. \end{aligned} \tag{2.2}$$

For the 3-D rectangular coordinate system, we have

$$\begin{aligned} x &= r \sin(\theta) \cos(\phi), \\ y &= r \sin(\theta) \sin(\phi), \\ z &= r \cos(\theta). \end{aligned} \tag{2.3}$$

### 2.1.5 Temporal Units of Measure

The clinker in the metric system is its temporal units. Indeed, both the metric and British system use the same basic measure of time, the second. The basic relationships and units of time are:

- 60 seconds (s.) = 1 minute (min.),
- 60 min. = 1 hour (hr.),
- 24 hours = 1 day,
- 365 days = 1 (calendar) year,<sup>4</sup>
- 365.25 days = 1 (sidereal) year.

Because of this we note that any coordinate system that includes time directly is inherently non-Cartesian.

### 2.1.6 Other Units of Measure and Coordinate Systems

As we have seen in our consideration of graphing in the previous chapter, we may define all manner of coordinate systems, such as  $x$  and  $f(x)$ . The units of measure of  $x$  and  $f(x)$  may be different, in which case the coordinate system cannot be Cartesian. Nonetheless, such coordinate systems are useful as we shall see as we proceed. In the main however, we shall primarily make use of spatial coordinate systems.

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<sup>4</sup>Every fourth year, unless the year is evenly divisible by 100 and not 400, has 366 days. Every year evenly divisible by 400 also has 366 days.

## 2.2 Some Basic Terminology

At this point it seems useful to discuss some basic terminology. We take as a given that the reader knows what an observation is, and expand from there. An **experiment** is one or more observations of some phenomena under controlled conditions. The observations of an experiment may be negative in the sense that nothing is observed and they may be indirect in that the phenomena may be sensed by some piece of equipment rather than directly with the human sense organs. Ideally control means that all aspects of the experiment are observed and quantitatively measured, and variation is limited to a single cause and a single effect.

A **model** is an informational representation of an observed phenomena. While a model may be linguistically descriptive, it is preferred that it be mathematical and/or logical in nature. A model is different from a simulation in the sense that a model is used in a simulation to produce numbers.

Hypothesis, theory, and law are statements of operation of one or more observable phenomena. The distinction among the three largely depends on how extensively they have been tested by experiment. An hypothesis will have the least testing and usually has a tentative aspect, being advanced to explain several observations. A theory will be extensively tested and shown to have broad applicability, but still need to be subjected to additional testing. A law will also be extensively tested and held to be generally applicable, but largely beyond further testing.

In modern usage, theory is the more commonly used term. Because all theory is seen as limited and transitory, the use of the term law has largely fallen away although legacy laws are still called such.

While the terms model and theory are sometimes used interchangeably, it is important to recognize that theory is somewhat the more definite in that it implies some explanatory cause and effect relationship.

## 2.3 Structured Variables

Up until now we have dealt with variables that only have enough structure to hold one number. Now we consider variables that may hold more than one number.

### 2.3.1 Matrices and Arrays

Consider a point in 3-D space. In a rectangular coordinate system, this point has coordinates  $(x, y, z)$ . Now, let us consider a variable, which we name  $[R]$  that can hold three numbers. We may use this variable to hold the coordinates of a point. Let us call this type of variable a **matrix** (plural matrices.) if it can contain either specific values or variables (as we have known them so far,) and an **array** if it can only hold specific numbers.

Now let us specify that a matrix (array) can have a structure. In particular, we may think of it as having dimensions, akin to a coordinate system, except that we will always arrange the structure like a rectangle, and the structure will be ordered by integer (counting) numbers. For example, if the structure is two dimensional, then we have a variable that is structured like a set of pigeonholes. If the variable is two dimensional we say that it has rows and columns, and if it is one dimensional, it either has only one row (but several columns) and is called a **row** matrix, or has only one column (but several rows) and is called a **column** matrix.

For most of what we shall be doing, we shall only be dealing with matrices with one or two dimensions. The matrix  $[R]$  thus is either a column or a row matrix with three components. We designate these components of  $[R]$  as  $R_i$ ,  $i = 1..3$ , although the number of components may be as large as desired. A matrix  $[S]$  that is two dimensional has components  $S_{ij}$ ,  $i = 1..I, j = 1..J$ . If  $I = J$  then we say that the matrix is **square**.

We may define addition of matrices by

$$[S] = [U] + [V], \quad (2.4)$$

which is meaningful only if the dimensions and number of components for each dimension match. In that case, we have the relationship

$$S_{ij} = U_{ij} + V_{ij} \forall i, j \quad (2.5)$$

which defines the addition operation for matrices and arrays. Subtraction is defined similarly.

Multiplication is more complicated, as we have to designate which dimension on each array is being multiplied. If this is not specified, then the last dimension of the first multiplying matrix is multiplied by the first dimension of the second multiplying matrix. We may designate this as

$$[S] = [U] \times [V], \quad (2.6)$$

although the symbols  $\bullet$  or  $*$  may be used, or occasionally no symbol at all. The student should pay careful attention to the convention used. The actual multiplication at the component level is defined by

$$S_{ik} = \sum_{j=1}^J U_{ij} V_{jk} \forall i, k. \quad (2.7)$$

Obviously, the number of components of the multiplied dimension of each matrix must be the same if the multiplication is to have meaning.

We may define the **transpose**  $[A]^T$  of a matrix  $[A]$  at the component level as

$$A_{ij}^T = A_{ji}. \quad (2.8)$$

This operation does not care about the number of dimensions or number of components per dimension. In effect, it turns a row matrix into a column matrix and *visa versa*.

Division is not defined for matrices or arrays. However, we do define the **inverse** matrix  $[A]^{-1}$  by

$$[A] \times [A]^{-1} = [A]^{-1} \times [A] = [1], \quad (2.9)$$

where  $[1]$ , which is called the **identity** matrix, is defined to be the matrix whose components have value one if all of the indices of that component are equal, and zero otherwise. Obviously this definition is most meaningful if the matrix is square. From this we may have the operation

$$[S] = [U] \times [V]^{-1}, \quad (2.10)$$

which is comparable to division. We note in passing that only certain matrices have inverses but the elaboration of the conditions of speciality is left for a later maths course.

### 2.3.2 Vectors

If we now associate a coordinate system with matrices, we may define a type of structured variable called a **vector**. While vectors may be transformed from one coordinate system to another, the operations on them and using them are usually specific to a given coordinate system. For our purposes we shall concentrate on the 3-D rectangular and spherical coordinate systems.

As a matter of definition, vectors are a subset of a class of mathematical objects called **tensors**. Vectors bear the same relationship to tensors that row and/or column matrices bear to matrices in general. Tensors are described by rank, which may be thought of as the number of dimensions. Thus, a vector is a tensor of rank one. Our previous variables, which could only hold one number, were tensors of rank zero, and are called **scalars**.

Tensors are mathematical transformation objects. A scalar transforms a point into itself. A vector transforms a point into another point. A tensor of rank two transforms a vector into another vector.

### Vector Composition and Arithmetic

A vector is usually written with a subscripted or superscripted tilde,  $\tilde{V}$  or  $\tilde{\tilde{V}}$ , or arrow,  $\underline{V}$  or  $\vec{V}$ . The latter is the older notation, the other three more

contemporary. A second rank tensor may be written similarly with either two tildes or a double headed arrow.

A vector may be written as a series

$$\underset{\sim}{V} = \sum_{i=1}^I V_i \hat{e}_i, \quad (2.11)$$

where  $I$  is the number of dimensions in the coordinate system, two if 2-D, three if 3-D, and the  $\hat{e}_i$  are called **basis** vectors. The  $\hat{\phantom{x}}$  which is called a "hat" in colloquial use, indicates that the vector is a **unit** vector.

We can define addition and subtraction in the usual way ala

$$\begin{aligned} \underset{\sim}{W} &= \underset{\sim}{U} + \underset{\sim}{V} & (2.12) \\ &= \sum_{i=1}^I U_i \hat{e}_i + \sum_{i=1}^I V_i \hat{e}_i \\ &= \sum_{i=1}^I (U_i + V_i) \hat{e}_i. \end{aligned}$$

There are two types of multiplication that we shall be concerned with, scalar and vector. These are colloquially referred to as the **dot** and **cross products**, respectively. The dot product is defined by

$$\begin{aligned} W &= \underset{\sim}{U} \bullet \underset{\sim}{V} & (2.13) \\ &= \sum_{i=1}^I U_i \hat{e}_i \bullet \sum_{j=1}^J V_j \hat{e}_j \\ &= \sum_{i,j=1}^{I,J} U_i V_j \hat{e}_i \bullet \hat{e}_j, \end{aligned}$$

where obviously the two vectors must have the same number of components (dimensions.) The **magnitude** of a vector is defined by

$$\begin{aligned} \left| \underset{\sim}{W} \right| &= \sqrt{\underset{\sim}{W} \bullet \underset{\sim}{W}} & (2.14) \\ &= \sqrt{\sum_{i,j=1}^{I,J} W_i W_j \hat{e}_i \bullet \hat{e}_j}. \end{aligned}$$

A unit vector is one that has unit magnitude.

In an orthogonal coordinate system, the basis vectors are orthogonal and unit, so

$$\hat{e}_i \bullet \hat{e}_j = \delta_{ij}, \quad (2.15)$$

where the  $\delta_{ij}$ , which is called the **Kronecker delta function**, is defined by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (2.16)$$

From this we may see that the dot product, equation 2.13 reduces to

$$\underset{\sim}{U} \bullet \underset{\sim}{V} = \sum_{i=1}^I U_i V_i, \quad (2.17)$$

and the magnitude, equation 2.14 reduces to

$$|\underset{\sim}{W}| = \sqrt{\sum_{i=1}^I W_i^2}.$$

The cross (vector) product is defined by

$$\begin{aligned} \underset{\sim}{W} &= \underset{\sim}{U} \times \underset{\sim}{V} \\ &= \sum_{i,j=1}^{I,J} U_i V_j \hat{e}_i \times \hat{e}_j. \end{aligned} \quad (2.18)$$

At this point our saying "x cross y into z" comes into play. Again, we must have an orthogonal system, from whence we define

$$\hat{e}_i \times \hat{e}_j = \begin{cases} \hat{e}_k; & i, j, k \text{ symmetric} \\ -\hat{e}_k; & i, j, k \text{ antisymmetric} \end{cases} \quad (2.19)$$

By this we mean that for a 3-D rectangular coordinate system where  $x, y, z$  are represented by the indices 1, 2, 3, any form of  $i, j, k$  that keeps the order of the coordinate indices ( $= 1, 2, 3; 2, 3, 1; 3, 1, 2$ ) is symmetric and any that does not keep the order (reverses a pair of index values) ( $= 1, 3, 2; 3, 2, 1; 2, 1, 3$ ) is antisymmetric. Any combination with two indices equal is zero.

We sometimes designate the basis vectors of a 3-D rectangular coordinate system as  $\hat{x}, \hat{y}, \hat{z}$ , or  $\hat{i}, \hat{j}, \hat{k}$ . The cross products are specifically

$$\begin{aligned} \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{x} &= -\hat{z} \\ \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{y} &= -\hat{x} \\ \hat{z} \times \hat{x} &= \hat{y} \\ \hat{x} \times \hat{z} &= -\hat{y}. \end{aligned} \quad (2.20)$$

As with matrices, vector division is not defined *per se*, but we can perform the operations

$$W = \frac{\underset{\sim}{U} \bullet \underset{\sim}{V}}{|\underset{\sim}{V}|^2}, \quad (2.21)$$

or

$$\underset{\sim}{W} = \frac{U \times V}{\underset{\sim}{|V|}^2}. \quad (2.22)$$

The basis vectors of the spherical coordinate system,  $\hat{r}, \hat{\theta}, \hat{\phi}$ , which are also orthogonal, may be given in terms of the rectangular coordinate system basis vectors as

$$\begin{aligned} \hat{r} &= \hat{x} \sin(\theta) \cos(\phi) + \hat{y} \sin(\theta) \sin(\phi) + \hat{z} \cos(\theta) \\ \hat{\theta} &= \hat{x} \cos(\theta) \cos(\phi) + \hat{y} \cos(\theta) \sin(\phi) - \hat{z} \sin(\theta) \\ \hat{\phi} &= -\hat{x} \sin(\phi) + \hat{y} \cos(\phi). \end{aligned} \quad (2.23)$$

The inverse relationships are

$$\begin{aligned} \hat{x} &= \hat{r} \sin(\theta) \cos(\phi) + \hat{\theta} \cos(\theta) \cos(\phi) - \hat{\phi} \sin(\phi) \\ \hat{y} &= \hat{r} \sin(\theta) \sin(\phi) + \hat{\theta} \cos(\theta) \sin(\phi) + \hat{\phi} \cos(\phi) \\ \hat{z} &= \hat{r} \cos(\theta) - \hat{\theta} \sin(\theta). \end{aligned} \quad (2.24)$$

Demonstration of these will be left to the student.

## Vector Functions

We may consider the coordinates of any coordinate system as variables. Thus, for the rectangular and spherical coordinate systems, we may have  $x, y, z$  or  $r, \theta, \phi$  as variables. We may thus consider scalar functions of any or all of these variables as  $f(x, y, z)$  or  $f(r, \theta, \phi)$ . Normally, we will not intermix coordinate variables of different systems although exceptions do occur. Further, it is perfectly acceptable to transform from one coordinate system to another, as we shall demonstrate shortly.

We may also consider vector (or tensor) functions as

$$\underset{\sim}{f}(\underset{\sim}{R}) = \sum_{i=1}^3 f_i(\underset{\sim}{R}) \hat{e}_i, \quad (2.25)$$

in 3-D space, where the  $f_i$  may be expressed in any useful coordinate system. Usually, the function is expressed in the same coordinate system as its variables. Thus, we would write

$$\underset{\sim}{f}(\underset{\sim}{R}) = f_x(R_x, R_y, R_z) \hat{x} + f_y(R_x, R_y, R_z) \hat{y} + f_z(R_x, R_y, R_z) \hat{z}, \quad (2.26)$$

or

$$\underset{\sim}{f}(\underset{\sim}{R}) = f_r(R_r, R_\theta, R_\phi) \hat{r} + f_\theta(R_r, R_\theta, R_\phi) \hat{\theta} + f_\phi(R_r, R_\theta, R_\phi) \hat{\phi}. \quad (2.27)$$



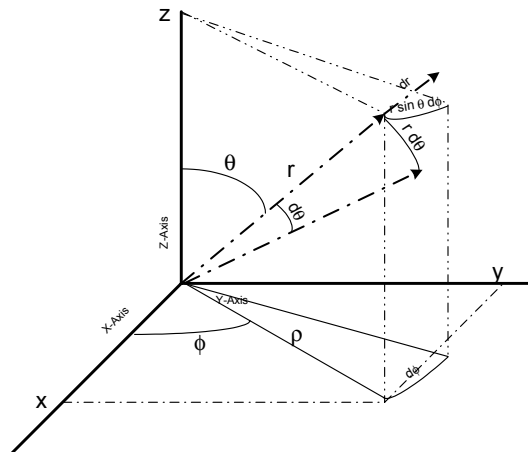
### 2.3.3 Vector Calculus

Arrays do not have a calculus, since as specific number holders they are, in effect, constants. Matrices in the sense that they are holders of variables, may have a calculus. The calculus of variable matrices however, is so similar to that of vectors that we shall not belabor both here, but concentrate on the calculus of vectors, which we shall use much more frequently. In most cases, the techniques of both finite and infinitesimal calculus are applicable, although if we do not qualify explicitly, we are usually using the infinitesimal version.

#### Differentials

Since we are dealing with a multi-dimensional system, we may have multi-dimensional differentials. For example, in the rectangular system, we have either two or three single dimensional differentials to use,  $dx, dy, dz$ . In 2-D, the area differential  $dArea = dx dy$ , but in 3-D can be any combination of three differentials that corresponds to the geometry of the situation at hand. For example, if we want to know an area perpendicular to the x-axis, then  $dArea = dy dz$ . In 2-D there is no volume differential, but in 3-D,  $dVol = dx dy dz$ .

The situation is somewhat more complicated in the circular and spherical coordinate systems since one (two) of the variables are angles. In this case, a bit of drawing comes in handy as in this figure.



Construction of the differential elements in the 3-D spherical coordinate system

In this case we see that while the differentials of each variable are  $dr, d\theta, d\phi$ , the differential (infinitesimal) lengths associated with them are:  $dr, r d\theta, r \sin(\theta) d\phi$ . The first is fairly obvious since it is itself a length. The second follows from rotating  $r$  through an angle  $d\theta$  giving us an arc of length  $r d\theta$ . The third follows from rotating  $\rho$  through an angle  $d\phi$ , which gives us an arc of length  $\rho d\phi$ . Since  $\rho = r \sin(\theta)$ , this is just  $r \sin(\theta) d\phi$ .

The volume differential is then obviously

$$\begin{aligned} dVol &= (dr) (r d\theta) (r \sin(\theta) d\phi) \\ &= r^2 \sin(\theta) dr d\theta d\phi. \end{aligned} \quad (2.28)$$

and again, the area element depends on what orientation of area we are interested in. The most common one considered in the spherical coordinate system is the area of a section of sphere. Under these conditions, the radius is fixed, so the area differential is just

$$dArea = \sin(\theta) d\theta d\phi. \quad (2.29)$$

If we make use of the (infinitesimal) exponential function, we may simplify these somewhat. One of the greatest secrets of math is the relationship

$$e^{i\psi} = \cos(\psi) + i \sin(\psi), \quad (2.30)$$

knowledge of which is usually reserved for senior or graduate courses in complex variables. And yes, the  $i$  in this case is  $\sqrt{-1}$ , so we are dealing with complex numbers.

If we now take the derivative

$$\frac{d}{d\psi} e^{i\psi} = i e^{i\psi}, \quad (2.31)$$

and substitute equation 2.30, we obtain

$$\frac{d}{d\psi} [\cos(\psi) + i \sin(\psi)] = i [\cos(\psi) + i \sin(\psi)], \quad (2.32)$$

where we have inserted the braces for clarity. Now, let us resolve the RHS of this equation, giving us

$$\frac{d}{d\psi} [\cos(\psi) + i \sin(\psi)] = [i \cos(\psi) - \sin(\psi)]. \quad (2.33)$$

We may now decompose this by equating separately real and imaginary parts of the equation since they are independent. This gives us

$$\begin{aligned} \frac{d}{d\psi} \cos(\psi) &= -\sin(\psi), \\ \frac{d}{d\psi} \sin(\psi) &= \cos(\psi). \end{aligned} \quad (2.34)$$

Let us now take the first of these two equations, and using the rare opportunity to treat differentials as simple variables, we may rewrite this as

$$d \cos(\psi) = -\sin(\psi) d\psi, \quad (2.35)$$

from which we may rewrite equation 2.28 as

$$dVol = -r^2 dr d\cos(\theta) d\phi. \quad (2.36)$$

This can be used handily on occasion, as we shall shortly demonstrate when we discuss angle units of measure. Note particularly the minus sign. As we have indicated, one of the few times we may treat differentials as simple variables is when we are **changing variables** and must relate the differential of the old variable to the differential of the new variable. Sadly, the whole matter of change of variable is an art form that we shall only show here and the student must await advanced math courses to see in comprehensive detail.

## Vector Derivatives

In the previous chapter we introduced the idea of a derivative of a function  $f$  of a single variable  $x$ , which we denoted as  $df(x)/dx$ . Now, we extend the concept to functions of several variables. If we have a function  $f(x, y, z)$ , then we may define the partial derivative of  $f$  with respect to  $x$ , denoted by  $\partial f(x, y, z)/\partial x$ , by

$$\frac{\partial}{\partial x} f(x, y, z) = \lim_{\Delta x \rightarrow 0} \left. \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x} \right|_{y, z \text{ fixed}}. \quad (2.37)$$

This is called the partial derivative because the function depends on more than one variable and thus a given partial derivative addresses only one part or variable.

From this we may proceed to the basic vector derivative operations. These are more structured than what we have seen in previous discussion largely because of the structured nature of the variables and/or functions.

**Gradient** The **gradient** is a derivative (sometimes called differential) operation usually performed on a scalar function. It can be performed on a vector but we shall not deal with such except for one special case, which follows subsequently.

We shall be using the symbol  $\nabla$ , called the 'del' for these operations. In the rectangular coordinate system, the form of del does not vary with application, again because all of the coordinates have the same dimensional form of length. This is not the case for the spherical coordinate system where two of the coordinates are angles.

In the rectangular coordinate system, the gradient of a scalar function is defined by

$$\tilde{\nabla} f = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) f. \quad (2.38)$$

In the spherical coordinate system the gradient of a scalar function is defined by

$$\tilde{\nabla} f = \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin(\theta)} \frac{\partial}{\partial \phi} \right) f. \quad (2.39)$$

As previously stated, one usually casts the function in terms of the appropriate (matching) coordinate system, in this case as either  $f(x, y, z)$  or  $f(r, \theta, \phi)$ , respectively.

**Divergence** The **divergence** is a derivative operation that is only defined on a vector (tensor) function. In the rectangular coordinate system, the divergence of a vector function is defined by

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{f} = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \bullet \underset{\sim}{f}, \quad (2.40)$$

which we see has the opposite effect of the gradient in that the divergence transforms a vector into a scalar while a gradient transforms a scalar into a vector. In the spherical coordinate system, the divergence is defined by

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{f} = \frac{1}{r^2 \sin(\theta)} \left[ \sin(\theta) \frac{\partial}{\partial r} (r^2 f_r) + r \frac{\partial}{\partial \theta} (\sin(\theta) f_\theta) + r \frac{\partial}{\partial \phi} f_\phi \right], \quad (2.41)$$

which we see by comparison with equation 2.39 has considerably different form. The divergence compares to the dot or scalar vector product.

**Curl** In the same fashion, the **curl** is a derivative operation that corresponds to the cross or vector product. In the rectangular coordinate system, the curl is defined by

$$\begin{aligned} \underset{\sim}{\nabla} \times \underset{\sim}{f} &= \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \underset{\sim}{f} \\ &= \hat{x} \left( \frac{\partial}{\partial y} f_z - \frac{\partial}{\partial z} f_y \right) + \hat{y} \left( \frac{\partial}{\partial z} f_x - \frac{\partial}{\partial x} f_z \right) + \hat{z} \left( \frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x \right). \end{aligned} \quad (2.42)$$

In the spherical coordinate system, the curl is

$$\begin{aligned} \underset{\sim}{\nabla} \times \underset{\sim}{f} &= \hat{r} \left[ \frac{\partial}{\partial \theta} (r \sin(\theta) f_\phi) - \frac{\partial}{\partial \phi} (r f_\theta) \right] \\ &\quad + r \hat{\theta} \left[ \frac{\partial}{\partial \phi} f_r - \frac{\partial}{\partial r} (r \sin(\theta) f_\phi) \right] \\ &\quad + r \sin(\theta) \hat{\phi} \left[ \frac{\partial}{\partial r} (r f_\theta) - \frac{\partial}{\partial \theta} f_r \right]. \end{aligned} \quad (2.43)$$

**Divergence of the Gradient** Because the gradient transforms a scalar into a vector, and the divergence transforms a vector into a scalar, the combination of the two can transform a scalar into a scalar. In the rectangular coordinate system this is

$$\underset{\sim}{\nabla} \bullet \underset{\sim}{\nabla} f = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f, \quad (2.44)$$

while in the spherical coordinate system it is

$$\nabla_{\sim} \bullet \nabla_{\sim} f = \frac{1}{r^2 \sin(\theta)} \left[ \sin(\theta) \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} f \right) + \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial}{\partial \theta} f \right) + \frac{1}{\sin(\theta)} \frac{\partial^2}{\partial \phi^2} f \right]. \quad (2.45)$$

A shortcut notation for the divergence of the curl is

$$\nabla_{\sim} \bullet \nabla_{\sim} = \nabla^2, \quad (2.46)$$

where  $\nabla^2$  is called the **Laplacian**.

**Curl of the Gradient** Similarly, we may calculate the curl of the gradient, transforming, in another way, a scalar into a vector. This derivative operation is

$$\nabla_{\sim} \times \nabla_{\sim} f = \underset{\sim}{0}, \quad (2.47)$$

where  $\underset{\sim}{0}$  is a vector all of whose components are zero. Note that this operation is independent of coordinate system. The vector  $\underset{\sim}{0}$  is called the **zero vector**.

**Divergence of the Curl** The last derivative operator we shall consider is the divergence of the curl. This is

$$\nabla_{\sim} \bullet \nabla_{\sim} \times f = 0, \quad (2.48)$$

which is also independent of coordinate system except for the form of the vector function, Note that this operation transforms a vector into a scalar.

**More on Partial Derivatives** It should go without much effort that if a function (scalar or vector) does not depend on a coordinate variable, then the partial derivative with respect to that variable is zero.

If we have

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad (2.49)$$

then all three of the (first) partial derivatives  $\partial f(x, y, z) / \partial x$ ,  $\partial f(x, y, z) / \partial y$ ,  $\partial f(x, y, z) / \partial z$  are nonzero. If we transform the function into the spherical coordinate system, we get

$$f(r, \theta, \phi) = r. \quad (2.50)$$

In this case,  $\partial f(r, \theta, \phi) / \partial r \neq 0$ , but  $\partial f(r, \theta, \phi) / \partial \theta$ ,  $\partial f(r, \theta, \phi) / \partial \phi$  are both zero since the function does not explicitly contain either angle coordinate.

Thus, we may write the gradient in the rectangular coordinate system as

$$\nabla_{\sim} f = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (\hat{x}x + \hat{y}y + \hat{z}z), \quad (2.51)$$

and in the spherical coordinate system as

$$\begin{aligned}\nabla_{\sim} f &= \hat{r} \frac{\partial}{\partial r} r, \\ &= \hat{r}.\end{aligned}\tag{2.52}$$

It may be useful in considering this for the student to compare equation 2.49 with equation 2.2.

### Vector Integrations

Just as there are several different vector derivative operations, there are several different vector integral operations. These are in addition to integrals that take on the geometry of the particular coordinate system such as

$$\begin{aligned}\int_V f(x, y, z) \, dx \, dy \, dz, \\ \int_V f(r, \theta, \phi) \, r^2 \sin(\theta) \, dr \, d\theta \, d\phi.\end{aligned}\tag{2.53}$$

The subscript "V" on the integral sign indicates that a particular volume, usually in the form of definite limits, will be specified but for now is treated as a variable. Area integrals also fall in this category. Technically these integrals are not vector integrals although the function may be a vector function so the distinction can become quite moot.

But we may also have integrals like

$$\begin{aligned}\int_C f \, dR_{\sim}, \\ \int_C f \bullet dR_{\sim}, \\ \int_C f \times dR_{\sim},\end{aligned}\tag{2.54}$$

where the subscript "C" on the integral sign signifies that the integration is to be performed on a path or **contour** that is to be specified.

**Gauss' Theorem** Gauss' Theorem, which we will state only by its mathematical form, is

$$\int_{S_{\sim}} f \bullet d\sigma_{\sim} = \int_V \nabla_{\sim} \bullet f \, dVol_{\sim},\tag{2.55}$$

where the subscript "S" signifies a surface (area) in or on the volume "V" specified by the vector  $\sigma_{\sim}$ . This theorem is a means of of interchanging a volume integral for a surface integral, or *visa versa*.

**Stokes' Theorem** Stokes' Theorem is expressed as

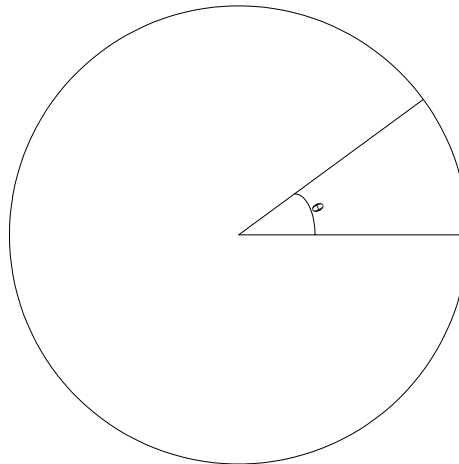
$$\int_S \nabla \times \mathbf{f} \cdot d\boldsymbol{\sigma} = \oint \mathbf{f} \cdot d\mathbf{R}, \quad (2.56)$$

where  $\boldsymbol{\sigma}$  is again a vector describing a surface. The integral  $\oint d\mathbf{R}$  indicates an integration along a closed path on the surface described by the vector  $\mathbf{R}$ . This theorem is a means of of interchanging a surface integral for a line integral, or *visa versa*.

## 2.4 Angle Units of Measure

There are two types of angles, **planar** and **solid angles**. The planar angles, which are usually just called angles are the ones we think of in the context of a right triangle, or a circular 2-D coordinate system. These (planar) angles are (can be) defined on a unit circle, that is, a circle whose radius is one unit (of something) long. (We don't care very much about what the unit is, as we shall shortly see.)

A unit circle has a circumference of  $2\pi$  units, and we define the angle of a full circle to be  $2\pi$  **radians**. So if we specify an angle  $\theta$  on the unit circle as shown in this figure,



Definition of Angle on a Circle

then we may see that the angle may be calculated as the length of the arc that the angle forms on the circumference of the circle divided by the radius of the circle. This is why we don't care very much whether we use a unit circle since we will be dividing a length (the arc length) by another length (the radius.) We also see why there are  $2\pi$  radians, since the circumference of the circle is  $2\pi$  times the radius of the circle.

When dealing with mathematical representations of angles, such as with the trigonometric functions, we must express angles in radians. When we do computations however, we may be able to use other units of measure. For example, most hand calculators allow us to specify the units of measure of angle, usually as radians, degrees, or grads. This is possible because the calculator does not use the trigonometric functions directly, but interpolates from a stored table of functional specific values.

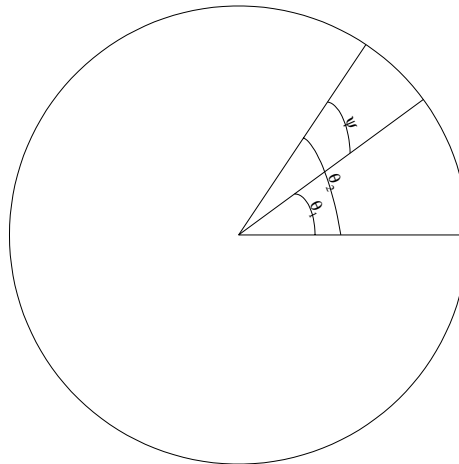
In the computational context then, we may specify other units of measure of angle. As we have indicated, the two most common such are degrees (360 to a circle) and grads (400 to a circle). This gives us the relationships between these and radians:

- $2\pi$  radians = 360 degrees = 400 grads.

We may also specify an angle  $\Psi$  as an integral

$$\begin{aligned}\Psi &= \int_{\theta_1}^{\theta_2} d\theta, \\ &= \theta_2 - \theta_1,\end{aligned}\tag{2.57}$$

as indicated in this figure.



Specification of an angle as the difference between two other angles

We recognize the differential  $d\theta$  as the infinitesimal of angle where  $r d\theta$  is the infinitesimal of arc length. The interpretation as angle follows trivially for the planar angle, but as we move to the solid angle, this integral provides a basis of comparison for the solid angle.

The surface of a sphere has an area of  $4\pi r^2$ , so if we divide this by the square of the radius, we get a total solid angle of  $4\pi$  **steradians**. For a lesser solid angle, we return to the angular surface differential for the spherical coordinate system as extracted from equation 2.36 as

$$dArea = -r^2 d \cos(\theta) d\phi.\tag{2.58}$$



In keeping with our previous, albeit trivial, planar angle definition, we may now define a solid angle  $\Omega$  in term of an integral over the surface of the sphere (of set radius) bounded by four (two pairs) of planar angles,

$$\begin{aligned}\Omega &= \frac{-r^2 \int_{\theta_1}^{\theta_2} d \cos(\theta) \int_{\phi_1}^{\phi_2} d\phi}{r^2} \\ &= [\cos(\theta_1) - \cos(\theta_2)] [\phi_2 - \phi_1],\end{aligned}\tag{2.59}$$

where we take  $\theta_1 < \theta_2$  and  $\phi_1 < \phi_2$  as sizes are defined in the coordinate system.

## 2.5 Trajectories

In the John Wayne movie "Hitari", there is a segment where the nerd character 'Pockets' (played by Red Buttons) has built a rocket to carry a net across a tree to entrap some monkeys. When he is asked about the placement of the rocket he replies that he has studied physics and knows about, among other things, "trajectors".

We have to assume here that the character has somehow lumped together the terms "vector", which we have already discussed, and "trajectory", which we are about to take up, and which, as we shall see, are related via vectors. The common definition of a **trajectory** is the path (of motion) taken by something over time.

### 2.5.1 The Trajectory of Physics

In a sense, trajectories are both a fundamental topic of physics, and a characteristic of how physics works, so this section will be a bit jumbled as we introduce both. In its most common usage, a trajectory is the path of something in 3-D space over time. If we think about this statement, we notice quickly that we are really talking about a four-dimension system with three dimensions being spacial (length units) and the fourth temporal (time units.) Strictly speaking, this four dimensionality is indeed the case, and we shall often draw graphs using the four dimensional coordinate values. Normally however, we shall speak of trajectories as being 1-D, 2-D, or 3-D in their spatial components only. (Some 3-D trajectories are really only 2-D - wait for it!) The exceptions will usually only come when we treat with Relativity where time does not behave parametrically.

What do we mean by **parametric time**? Strictly put, it means that the rate of change of time is constant, that time always changes at the positive

rate of one second per second, never 0.9 or 1.1. Because we are imbedded in time, we have no way to independently measure time and the assumption of constant time rate is the simplest assumption that we may make. (Thus, parametric time is basically the simplest assumption that we can make about time to handle it mathematically. We shall discuss the importance of simplicity shortly) We may have two observers measure a segment of time independently, with identical time measuring devices, and except under relativistic conditions, they will both measure the same amount of time, consistent with the error of measurement. In a relativistic environment, where one observer is accelerated, then they will measure different amounts of time, but we always take one of the observers to be the basis of comparison since we are embedded in time and cannot measure it independently.

When we are not dealing with a relativistic situation, and that will be the unexpressed norm in this book, we take everyone to be embedded in the same stream of time and we measure the progress of time by observing some physical process.

The same also applies in making spatial measurements because, again, we are embedded in space. We may not step outside space but we may make comparative measurements between two observers although, again, we must take one of the observers to be the basis of comparison. Except under relativistic conditions, the two should observe the same measurement.

Now we get holistic for a moment. All conditions are relativistic if Relativity theory is correct. However, sometimes the situation is such that the effect of relativity is quite small and we can safely neglect it in most observations. These conditions are those under which we say that relativistic conditions do not apply.

It is at this point that we return to our earlier admonition that "To fully and accurately learn physics, the ideas and concepts of physics must first form a basis on their own for viewing the world." Physics is a subject of some complexity, and one of the tricks that we play to make it more practicable is to introduce approximation as a disciplined tool. Simply put, when an approximation may be made that will not cause significant (but quantifiable) inaccuracy, but will simplify the modeling and understanding of the situation, then we may make use of the approximation. In this case, if the acceleration environment is not strong and velocities are not too high, then we may safely treat the situation without concern for Relativity.

This is hard to grasp, and not just because we are dealing with concepts that we have not discussed such as velocity, acceleration, and relativity. Nonetheless, this is necessary, that we learn to understand and think about physics in the context of the physics itself, and not in the context of the environment we have grown up in. In the latter context, the response would be that stating that we cannot independently measure time and space would be viewed as rubbish because we have clocks and yard sticks. Nonetheless, if one thinks about it, one realizes that if space is non-linear (e.g.,) then the non-linearity applies equally well to the yard stick as to the thing we measure with it. Thus, to proceed

with measurement, and physics, we have to think about this in terms of physics first, and then later in more normal terms of human experience.

We have to make these assumptions about the linearity of space and time so we may develop mathematical models of how things behave in space and time. The assumption of linearity is the simplest such that we may make, and it does break down under observation. That is where Relativity comes in, for those situations when space and time are not linear.

There are two points we need to consider here. The first is that simple models provide a basis for the development of more complicated models, which reduce to the simple models under the appropriate conditions and the mathematical representation of those conditions. In this sense, the models, which are the body of physics, are self-consistent. This does not "prove" that the physics is correct, but it does demonstrate an internal agreement with what we observe of the universe.

Second, there are observations that the models do not agree with by a margin that exceeds the accuracy of our measurements (observations.) This indicates that the models of physics are not complete and there is yet work to be done in developing new models.

This is the nature of science, of which physics is a part. The goal of science is understanding of the universe. Science is done by observation and experiment, which are used to build theory, which is used to make predictions or develop experimental tests of the theory, which are tested by more observation and experiment, which is used to build theory, through confirmation and embellishment or originality, which ..... in a cycle that continues until all possible observations and experiments have been made and theory developed. In this regard, science (and physics) is never perfect because it is never fixed, but is constantly changing due to the interplay between experiment/observation and theory.

In a similar sense, this interplay may be viewed as the trajectory of physics where, over time, we move back and forth from experiment to theory to experiment ....

So fundamental is this that we often distinguish between physicists who are primarily concerned with experimentation and those primarily concerned with theory. The distinction is somewhat artificial in that it arises out of the complexities of doing experiments, which requires specialized knowledge and skills in the equipment used to perform the experiments, much of which must be created or built from scratch, and the collection of the data in the experiments, and of doing theory, which requires specialized knowledge and skills of the mathematical and other methods of doing theory. The boundary between the two, of comparing theory to experiment and experiment to theory, is shared by **experimentalists** and **theoreticians**. Both experimentalists and theorists are physicists, but differ in what tools they use on a daily basis much as carpenters who build houses and carpenters who build furniture both do carpentry but use different tools on a daily basis.

## 2.5.2 Trajectories in Physics

Moving now from our discussion of the nature of the practice of physics to a matter of its content, one of the fundamental concepts of physics is that of the trajectory. While we may mean by trajectory the path of any variable over time, the most common use we shall make of the term is still that of the motion of something in space over time.

This is what much of so-called basic physics is all about; it is often from the observation of the behavior of things over time (in a quantitative manner) that we can come to understand how they work. This is the basis of what is often called **mechanics**, which has to do with the behavior of thing under the application of forces (which we have not yet defined, but wait for it!)

Usually we are concerned with the trajectories of material things, that is things that are comprised of matter. In this case, we normally are interested in the vector (3-D space) entities:

- the position;
- the velocity; and
- the acceleration of the thing.

All these are normally taken, at least ultimately, to be functions of parametric time.

The **position** vector, often signified by the symbols  $\tilde{R}$  or  $\tilde{r}$ , is nothing more than the functional representation of the coordinates of the point where the "thing" is at a specific value of time. Thus, we may have

$$\tilde{R} = \hat{x}x + \hat{y}y + \hat{z}z, \quad (2.60)$$

in a rectangular coordinate system, or

$$\tilde{R} = \hat{r}r + \hat{\theta}\theta + \hat{\phi}\phi, \quad (2.61)$$

in a spherical coordinate system, where the coordinates  $x, y, z$  and  $r, \theta, \phi$  are functions of time  $t$  and hence  $\tilde{R}$  is a function of  $t$ .

The **velocity** of the thing, sometimes signified by  $\tilde{V}$  or  $\tilde{v}$ , is simply defined by

$$\tilde{V} = \frac{d}{dt} \tilde{R}, \quad (2.62)$$

that is, as the (first) time derivative of the position. Similarly, the **acceleration**,  $\tilde{A}$  or  $\tilde{a}$ , is simply defined as

$$\begin{aligned} \tilde{A} &= \frac{d}{dt} \tilde{V} \\ &= \frac{d^2}{dt^2} \tilde{R}. \end{aligned} \quad (2.63)$$

By simple inversion of these definitions, we may write integral equations for the velocity and the position of

$$\begin{aligned}\tilde{V}(t) &= \tilde{V}(t_0) + \int_{t_0}^t \tilde{A}(t') dt', \\ \tilde{R}(t) &= \tilde{R}(t_0) + \int_{t_0}^t \tilde{V}(t') dt',\end{aligned}\tag{2.64}$$

where  $t_0$  is some specific time, nominally when we begin the observation. We can combine these two and write one integral equation

$$\begin{aligned}\tilde{R}(t) &= \tilde{R}(t_0) + \int_{t_0}^t \left[ \tilde{V}(t_0) + \int_{t_0}^{t'} \tilde{A}(t'') dt'' \right] dt' \\ &= \tilde{R}(t_0) + \int_{t_0}^t \tilde{V}(t_0) dt' + \int_{t_0}^t \int_{t_0}^{t'} \tilde{A}(t'') dt'' dt',\end{aligned}\tag{2.65}$$

where we have carefully renamed the first (acceleration) integration variable to avoid confusion (yes, it really does - try it!) Since  $t_0$  is a specific value, and thereby  $\tilde{V}(t_0)$  must be as well, we may treat these as constants of integration, and simplify this equation as

$$\tilde{R}(t) = \tilde{R}(t_0) + \tilde{V}(t_0) (t - t_0) + \int_{t_0}^t \int_{t_0}^{t'} \tilde{A}(t'') dt'' dt'.\tag{2.66}$$

## Speed

The term **speed** is often used with respect to trajectories. Sadly, the term has several definitions that are not always precisely associated with their specific use.

The most common definition of speed, often designated by  $S$  or  $s$ , is the (instantaneous) magnitude of the velocity

$$s = \left| \tilde{V}(t) \right|.\tag{2.67}$$

Another common definition of speed is

$$s = \left| \frac{1}{\Delta t} \int_{t-\Delta t}^t \tilde{V}(t') dt' \right|,\tag{2.68}$$

which is technically an averaged (over some time interval) speed. Note that this is not the same as either

$$\frac{1}{\Delta t} \int_{t-\Delta t}^t \left| \tilde{V}(t') \right| dt',$$

or

$$\sqrt{\frac{1}{\Delta t} \int_{t-\Delta t}^t \tilde{V}(t') \bullet \tilde{V}(t') dt'}.$$

A third common definition is

$$s = \left| \underset{\sim}{V}(t) \bullet \hat{d} \right|, \quad (2.69)$$

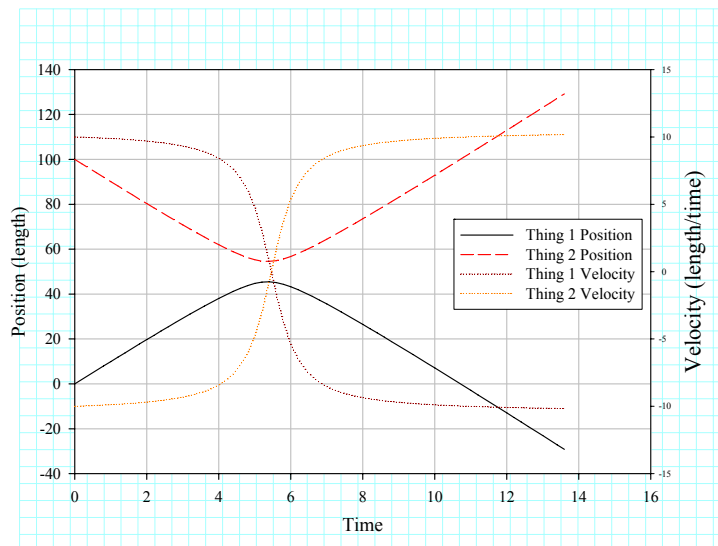
which is technically the component of projection along the direction represented by the unit vector  $\hat{d}$ . In this case the notation  $| \cdot |$  indicates absolute value since the quantity inside it is a net scalar. This type of speed is that measured by a Doppler radar such as is used by constables to determine the "speed" of automobiles.

Note that these three speeds do not generally have the same values. The first and second definitions are the same result only if  $\underset{\sim}{V}(t)$  is a constant, or by happenstance. The first and third are the same only if

$$\underset{\sim}{V}(t) \times \hat{d} = 0.$$

## Graphs of Trajectories

Before concluding this somewhat ambiguous discussion of trajectories, we want to examine a couple of graphs of trajectories. The first such is of two things moving in 1-D (space) towards each other. The two things do not like each other and "repel" each other. For the moment, we must leave the nature of this repulsion undescribed, but we display the trajectory, at least the position and velocities (which are 1-D vectors and can thus be displayed on a single planar graph) in this figure.

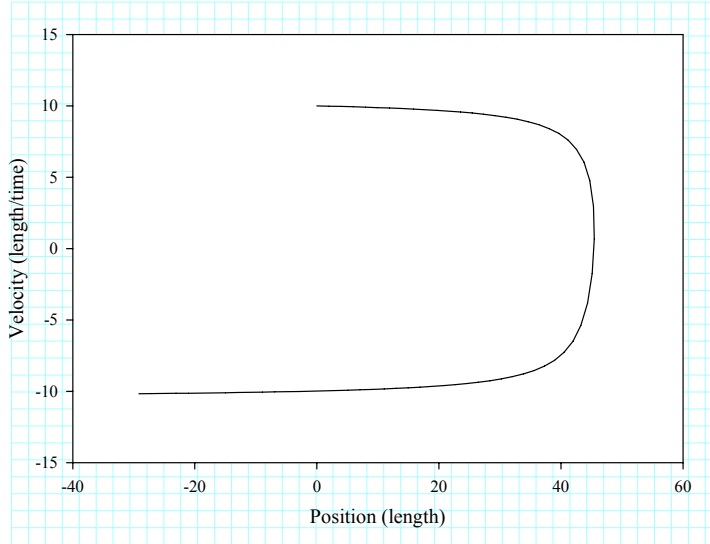


Trajectories of two things that repel

From this we may see that the positions (solid and long dashed lines) come together and then move apart along the direction they entered. We see that the velocities (dotted lines) start with one positive and one negative and end the

opposite, with the velocities changing faster the closer the things are together (the difference in the position vectors.)

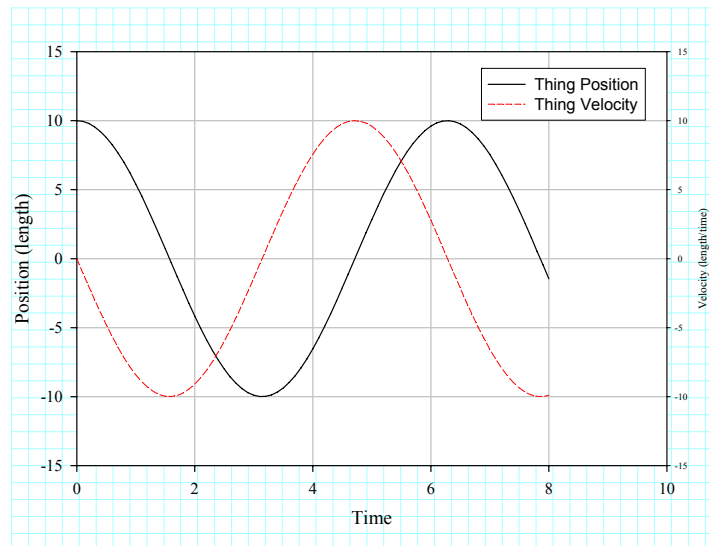
If we now plot the velocity of the first thing versus its position, what is sometimes called a **phase plot** or **graph**, in this figure,



Plot of Thing 1 velocity versus its position

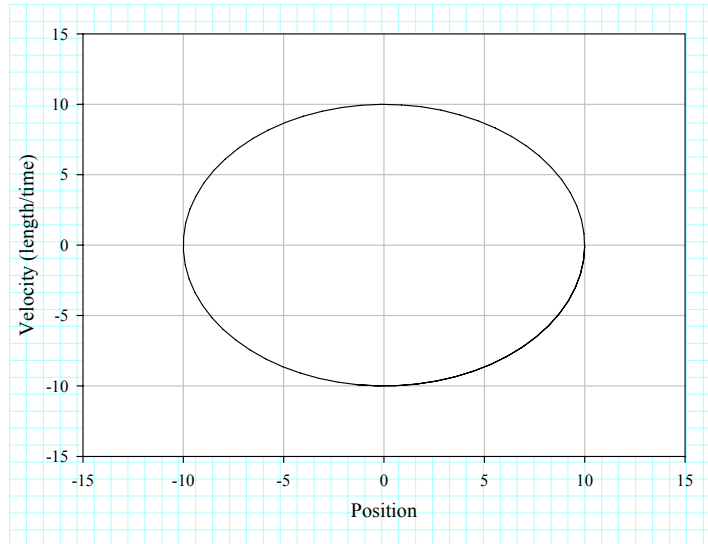
we see that the curve, which is a representation of the trajectory, is **open**, that is, it does not close back upon itself.

Now, let us repeat this consideration of a thing that oscillates like the bob of a pendulum clock. The position and velocity are given in this figure.



Position and velocity of an oscillating thing versus time.

In this we see that the motion, both position and velocity are bounded. This boundedness is clearer in this figure,



Velocity of oscillating thing versus its position

where we see that the curve closes over itself. We thus call this type of trajectory a **closed** trajectory.

We need to note something special here. The cycles of the trajectory overlay each other so that in this plot we see only one line. This one-linedness is not the general situation with a closed trajectory. Rather, a trajectory is closed if it is bounded; that is, nowhere does the curve go off to infinity. If we examine the curves of the open trajectory (two things that repel) above, we see that we can extend the position curves back to positive and negative infinity, respectively.

We shall elaborate further on the nature of trajectories and of openness and closedness as we proceed. For now, this is enough for our consideration.

## 2.6 Frequency Units of Measurement

There is a bit of ambiguity in the units of frequency that arises from the nature of mathematics. As such, special care is called for when we deal with it.

A closed trajectory is repetitive. By this we mean that the nature of the motion of the thing repeats itself. This repetition, as we have noted, is not always a strict repetition of position. When it is, repetitions of the trajectory overlap each other on a phase graph as in the previous section. Sometimes however, repetition may not be characterized by a position but by some other aspect. Common aspects are the points of closest approach to and farthest departure from the center of the graph (or center of the trajectory.)

Regardless, a closed trajectory repeats itself, and each repetition may be called a cycle. Each cycle takes the same amount of time and this amount of time is called the **period**  $T$  of the trajectory. Any activity that is regularly



repeated in this manner is said to be periodic, which simply means that it has a period. Obviously, the unit of measure of a period, since it is a time, is seconds.

The **frequency**  $f$  is defined simply to be the inverse of the period,

$$f \equiv \frac{1}{T}. \quad (2.70)$$

The unit of measure of frequency, which dimensionally is inverse time, is the Hertz (Hz), which is named after a physicist who did considerable research on electromagnetic waves. The quantity  $ft$  is just the number of cycles (since  $t = 0$ ) that the trajectory has gone through.

This leads us to the ambiguity. The mathematical representation of certain trajectories uses transcendental functions, usually sine and cosine. This is equivalent to representing a single cycle as the circumference of a unit circle. When this representation is used, we have to deal with what is called an **angular frequency**  $\omega$  which is defined as

$$\begin{aligned} \omega &= 2\pi f \\ &= \frac{2\pi}{T}. \end{aligned} \quad (2.71)$$

Care must be taken to distinguish between the two frequency definitions lest an error of a factor of  $2\pi$  result. This may be difficult but is necessary. It may be helpful to think about whether one is counting cycles (frequency) or a location on the trajectory (angular frequency.)

## 2.7 States

One of the ways in which we describe things both mathematically and physically is by what are often called **states**. This use of the word does not refer to a country or a part of a country, but is used in the sense that we associate the word with some observable characteristic. For example, it is common in the English language to say that someone is in a state of agitation or depression.

For example, we have characterized trajectories as being open or closed. We may intend this to say that a thing whose trajectory is closed is in a **bound state** because it is constrained to a repetitive path. Similarly, a thing whose trajectory is open may be said to be in a **free** or unbound state because it is not so constrained.

In a like manner, we may describe a penny as having two states when it is lying on a surface (if we discount or exclude the situation where it is balanced on its rim) - heads or tails depending on whether Lincoln's head or memorial is observable. Alternately, we may say that a thing in motion is in a state of motion that is characterized by the details of the trajectory, in particular its

position and velocity. Indeed, we sometimes use the term **state vector** as a synonym for trajectory or a position on a trajectory.<sup>5</sup>

From this, we may infer that states may be continuous or discrete, quantified or qualified, but in some manner always descriptive of the physical situation. Another way of saying this is that the state of a physical thing is observable.

## 2.8 Systems

From time to time, we shall use the term system to refer to something. As with many of the terms that we use in physics, we will use it somewhat ambiguously, largely because we will use slightly different definitions that we are not always precise about explicitly stating.

In general, a system may be only a single thing or a collection of things that may be treated as either one thing or several things. What makes these **systems** is that we may distinguish them from the rest of the universe on the basis of one or more observables.

In most cases, the quality that defines the system will be one or more states. For example, the coins in my pocket may comprise a system in that they all share the same states of heads and tails, and they are distinguished from the rest of the universe by being in my pocket.

Alternately, we talk about the Solar system which is distinguished from the rest of the universe by being the collection of all things in orbit (bound trajectory) around Sol.

The system concept is primarily important in terms of how it influences the nature of science and physics, in our particular interest. The general approach in physics may be described by what is often known as the "systems assumption", which is that systems are comprised of subsystems and the observable behavior of the system may be described by the mechanics of the subsystems by themselves and in interaction with each other and the rest of the universe.

There are two important pieces of this assumption. One is that the system is divisible, that it is not so complicated that it must be considered solely as an entity. This is actually an extension of how we defined system in the first place and it amounts to assumption that holism can either be decomposed or adequately approximated by decomposition.

The second important piece is that we first consider the behavior of a subsystem by itself, and then consider its interaction with the other subsystems and the rest of the universe. This gives an order to the introduction of complication into the consideration based on how closely the model/theory agrees with observation.<sup>6</sup>

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<sup>5</sup>One common usage of state vector is the combination of position and velocity vectors.

<sup>6</sup>In some systems theories, the definition of the thing that distinguishes the system from

In effect, this is an application of Ockham's razor, which variously states that "Plurality should not be posited without necessity", or "Entities are not to be multiplied beyond necessity." This rule is named for William of Ockham (1285-1347), who popularized it. It was introduced into physics largely by Galileo Galilei (1564-1642) who performed important experiments and developed early theory of mechanics and cosmology.

The basic idea of Ockham's razor is that if presented with two theories that explain observed phenomena, the simpler of the two is to be preferred. A somewhat terser contemporary statement of Ockham's razor, which often is sadly separated from its analytical aspects, is "Keep It Simple, Stupid!" expressed as the acronym "KISS".

In some usages of the system model, the terms input and output are used as synonyms for cause and effect.

We shall elaborate further on the concept of system as we proceed.

## 2.9 Randomness

In physics we often speak of different phenomena as being random or deterministic. The latter is somewhat easier to see initially. We say that something is **deterministic** if there exists a perfect **cause and effect** relationship. (In programming terms, this means an exact IF..THEN relationship.) By this we mean that if some action is taken (the cause), some result always occurs (effect.) If I am in a gravitational field and I hold a coin between my thumb and first finger and out from my body, then when I loosen my grip (the cause or action), the coin drops to the floor or ground (the result or effect.)

So long as I stay in a gravitational field, and I otherwise always have the same causative action, the coin will always fall. Indeed, it does not matter which hand I use, or what type of coin I use, or even (within a limit) how high above the ground I am. We thus say that the fall of the coin is deterministic.

Now let me curl my fingers and raise my thumb as I did earlier in discussion coordinate systems. I lower my thumb to rest it loosely in the crook of the bend of the second joint against the first finger about half way down the thickness of the finger. Now I rest the coin on top of the first finger centered on the second joint so that the edge of the thumb is underneath the coin, Now I flick the coin into the air and observe the state of the coin when it comes to rest (head/tails.) Now I repeat this several times, say ten or so, each time observing the state of the coin.

In this case, the coin is either in the head state or the tail state but not exclusively in either. If I count the number of times the coin is in each state then after a while the numbers should be close to each other but will not usually

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the rest of the universe is the nature of interactions within the system versus those between the system and the rest of the universe.

be the same. We say that under these circumstances the state of the coin is **random**, that is, it is in one or the other of the two states that it may occupy, but we cannot say definitely which it will be in before we flick the coin. We may refer to such phenomena as being either random or **stochastic**.

In such cases, we see that the cause and effect relationship is not perfect in the sense that we take some action and while the result is usually bounded in some sense, it is not strict, nor may it be strictly predicted before the fact. Indeed, in many stochastic situations, we may have no idea of the actual cause, or the cause may be too complex to characterize, so we get an effective situation of effect without definite cause.

### 2.9.1 Probability and Statistics

In dealing with stochastic phenomena we make use of two related bodies of information, probability and statistics. The relationship of the two is much like that in physics in that probability corresponds (rather loosely) to theory while statistics corresponds to experiment.

We use the term probability somewhat ambiguously to mean both a quantity and a theory or body of theory. Unfortunately, this ambiguity is intensified in that we have to define both in terms of experiment. The quantity **probability** is usually defined in two different ways that are not quite the same under all conditions. The first of these, which is sometimes called the **classical** definition, is to take a population (collection) of  $N$  things. These  $N$  things are identical in that while each may occupy any one of  $J$  states, they all share the same  $J$  states. We then observe, as nearly simultaneously as possible, which of the  $J$  states each of the  $N$  things occupies, and total these up as the number  $n_j$  of things in each of the  $J$  states. The probability that a thing is in the  $j^{\text{th}}$  state is then just

$$P_j \equiv \frac{n_j}{N}. \quad (2.72)$$

The other definition, which is called the **frequency** or frequentist definition, is to observe a single thing for a long time  $T$ ,<sup>7</sup> and observe the total time that the thing is in the  $j^{\text{th}}$  state,  $t_j$ . Then the probability that a thing of this type is in the  $j^{\text{th}}$  state is just

$$P_j \equiv \frac{t_j}{T}. \quad (2.73)$$

Often, we should like these two definitions to be the same, that is, that the probabilities are the same (specific values) for both definitions. We say that when a system (collection of things) has the same classical probabilities and frequency probabilities the system is **ergodic**, and that its distribution and/or statistics are **stationary**. (We may sometimes refer to this collection as an **ensemble**.)

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<sup>7</sup>Although we are using the same symbol, this is different, in general, from a period.

Stationary has a special meaning here. To discuss it we must introduce the idea of inner and outer time scales. The **inner time scale**, which we may call  $t_i$  for convenience, is the time that we spend making an observation of the system. If we use the classical definition method, then  $t_i$  is quite short, but if we use the frequentists definition method,  $t_i = T$ . The **outer time scale**,  $t_o$ , is the time between successive observations (measured from start to start or finish to finish, but consistently.) Obviously we must have  $t_o \geq t_i$ , but in general we want  $t_o > t_i$ , and often  $t_o \gg t_i$ , but not always. Then after many observations, many  $t_o$ , if the  $n_j$  or  $t_j$  are always the same, consistent with the error of observation, then we say the distribution (statistics) are stationary. (We discuss one measure of sameness under moments.)

Many time dependent processes are not stationary. Obviously, any system that has one or more **accumulating** states does not exhibit stationary behavior. An accumulating state is one that is permanent once it occurs. A thing in an accumulating state stays in that state.

This leads us to the concept of a Markov system. We shall use a somewhat restrictive definition here because this is how a Markov system is most commonly used in physics. The student is warned that other, more general, definitions of Markov systems exist.

If we have a thing that is known at time  $t$  to be in state  $j$ , we say that the thing is **Markovian** (or part of a **Markov** system) if the probability that the thing is in state  $j'$ , which may include  $j$ , at time  $t'$ ,  $t' > t$ , depends only on which state it is in now, but not previously. Thus, the state of the thing  $j''$  at time  $t''$ ,  $t'' < t$ , does not enter into the probability of what state the thing is in at  $t'$ .

We sometimes say that Markov systems have no memory of any previous state; they are "aware" only of the current state and forget it as soon as they change states, possibly to the same state. Being Markov and stationary are independent of each other.

As a matter of terminology, if we repeat some random process and observe the resulting state, each repetition is usually called a **trial**. This terminology is most natural to the frequency definition. If we observe the state of a member of a population (an ensemble), then this is often referred to as a **sample**, and the number of samples we take is called the **sample size**. When we are dealing with a sample size less than the population size, then there is some concern over whether the samples have maintained the random nature of the population.

If we know a population size, and predict how many things occupy each state, then we make use of probability theory although we may commonly refer to this as simply probability. The realm of statistics is to take the actual observations of the states and reconcile them with or develop theory. Often, we refer to the actual measurements and/or the analyses, as the **statistics** of the system.

Probabilities are always associated with observable states. These states may be either discrete or continuous, from which we describe their probability characterization as being either discrete or continuous. For our purposes, the

primary distinction will be whether we must use finite or infinitesimal mathematical techniques.

## 2.9.2 Random Variables

In this regard we may think of a state as a variable regardless of whether it is random or deterministic. We may refer to it as a random variable or a deterministic variable. As such, we talk about the variable having some particular value that may be a counting number if the states are discrete, or a real number if the states are continuous.

For example, a coin has two states: 1 and 2; that correspond to heads and tails (or *visa versa*.) A position in space may have an infinity of states and thus be continuous. In such cases we usually talk about the range of the states, which may be bounded or not. We recognize that this introduces an ambiguity. For a system of discrete states, a finite number of probabilities are used and thus definite probabilities exist. Continuous states do not have such definite probabilities since a line (e.g.,) consists of an infinite number of points, each of which should have a discrete probability but cannot because then the sum of the probabilities would itself be infinite. Instead we use the fiction of continuous mathematics to define probabilities on (e.g.,) segments of the line. We use this to make our lives easier and the ambiguity will resolve itself subsequently.

For a set of discrete states  $\{i : i = 1..I\}$  (states "i", where "i" goes from one to I), we may associate a probability  $P_i$  with each state. For continuous states  $x$ ,  $a \leq x \leq b$  (alternately we may write  $\{x | x \subset [a, b]\}$  to mean "x" such that "x" is contained on the bounded interval between "a" and "b", which we may shorten to  $\{x | [a, b]\}$  when the meaning would not be too ambiguous), we associate a probability function,  $P(x)$ , defined mathematically by

$$P(x) = \int_a^x p(y) dy, \quad (2.74)$$

by which we mean that the probability of "x" is actually the probability that the state of the system is no greater than "x". The function  $p(y)$  is referred to as the **probability density function** (or pdf), while  $P(x)$  is more technically correctly referred to as the **cumulative distribution function** (or cdf.) Notice that for a finite interval of  $x$ ,  $[x, x + \Delta x]$ , we may easily define a probability that the state is in the interval by  $P(x + \Delta x) - P(x)$ .

Regardless of whether we have a discrete or continuous (infinitesimal) system, probability is conserved. That is,

$$\sum_{i=1}^I P_i = 1, \quad (2.75)$$

for discrete systems, and

$$\int_a^b p(y) dy = 1, \tag{2.76}$$

for continuous systems. The latter implies that  $P(b) = 1$ , and  $P(a) = 0$ , which is normally the case. In some instances it may be useful to reverse the order of the interval and hence the convention.

In general, we may talk about a system (thing) being in a particular state only if the state is part of a discrete set of states, Otherwise we may only talk about a system being in a subrange of states if it has a continuous set of states.

**Remark 4** *We said initially that it was easier to deal with determinism than stochasticism. Now we want to briefly consider that physics indicates that determinism is a special form of stochasticism. By this we mean that the universe is inherently random but under certain special conditions, which are not uncommon, the physics may constrain the states in such a way that they behave in an essentially deterministic fashion.*

### 2.9.3 Moments

Let us now consider some developments of the idea of randomness. First, consider that we have some process that is characterized by some random variable  $x \subset [a, b]$ . If we are dealing with probability theory we should have some probability density function associated with this random variable,  $p(x)$ . If we are dealing with statistics, then we should have some set of measurements  $\{x_i : i = 1..I\}$  of  $x$ . In either case, we are interested in some related concepts.

If, for example, we are dealing with measured data, we may be interested in determining the behavior of the random variable. That is, we may want to infer, at least approximately, what  $p(x)$  is.

If, on the other hand, we are interested in predicting the outcome of experiments, then we may want to estimate or approximately compute what the measured data should look like.

Because the process is random, there is no simple deterministic way that we can predict the individual values of  $x_i$  from  $p(x)$ , nor is there any simple deterministic way that we may arrive at  $p(x)$  from the  $x_i$ . In these considerations we must account for the randomness, the stochastic nature, of the process.

In this sense, if we pick, by some method, a set of values  $\{x'_i\}$  from  $p(x')$ , then we should not expect a close degree of agreement between the  $x'_i$  and the  $x_i$  on an individual basis. We should however, expect that their behavior as collections  $\{x'_i\}$  and  $\{x_i\}$  should have some degree of agreement. In probability and statistics one such measure of comparison is moments, in reference to the pdf of the process.

For a continuous distribution (random variable,) the **moment** (of order  $n$ ) is defined by

$$\langle x^n \rangle \equiv \int_a^b x^n p(x) dx. \quad (2.77)$$

For a discrete random variable, the moment is

$$\langle x^n \rangle \equiv \sum_{j=1}^J x_j^n P_j. \quad (2.78)$$

The equivalent quantity in statistics is

$$\langle x^n \rangle \equiv \frac{1}{I} \sum_{i=1}^I x_i^n. \quad (2.79)$$

The symbol  $\langle \rangle$  is sometimes reserved to indicate an ensemble average, that is, an average over a population, collection, or ensemble. This definition corresponds to the classical definition of probability. When this distinction is made one talks about a time average of the form

$$\bar{x}^n(t) = \int_{t_{\min}}^{t_{\max}} x^n(t) p(t) dt. \quad (2.80)$$

This is, of course, a functional average, and for ergodic systems the two should be equivalent.

At this point a bit of diversion for explanation is called for. If we compare equations 2.78 and 2.79 we note that we have two sets  $\{x_j : j = 1..J\}$  and  $\{x_i : i = 1..I\}$  where the first set is the set of states' particular values, while the second is the set of observations' particular values. If we use our coin as an example, the  $x_j$  are the numeric values corresponding to heads and tails, while the  $x_i$  are the individual measurements or observations of particular trials or samples. Thus while the values of the  $x_j$  and the  $x_i$  are the same, either one or two, they have very different meanings. We also note that, of necessity,  $I > J$ , although often,  $I < N$ , the population size. If we now take the  $P_j$  and multiply by  $N$  (and round the results to integers - zero decimal place real numbers,) then these values should correspond (approximately) to the number of times the set of  $x_i$  have the particular value of the  $x_j$ . In other words, and following from the definition of probability, the number of times a particular state value occurs in a set of trials or samples is directly related to the probability of that state.

In statistics, the difference between a set of samples and a population is made if the sample size is less than the population. This applies to the moments. A particular moment, called the variance, is defined slightly differently for a population and for a sample (of size less than the population.) The first moments, called the **mean**, is the same regardless. The basic definition of the **variance** is

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2, \quad (2.81)$$



and its square root,  $\sigma$ , is called the **standard deviation**, which is normally the positive root. (There are exceptions in physics due to the mathematical form.)

The mean is defined identically for both sample and population, but the variance for a population is

$$\sigma_{population}^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \langle x \rangle)^2, \quad (2.82)$$

while the sample variance is

$$\sigma_{sample}^2 = \frac{1}{I-1} \sum_{i=1}^I (x_i - \langle x \rangle)^2. \quad (2.83)$$

The notable difference is that the denominator is reduced by one. This difference follows from the fact that the sample represents a subset of the population and there are many subsets possible. Many basic textbooks will claim this difference is because of the smaller size of the sample variance but this is an effect erroneously labeled in such cases as a cause.

It is also convenient that we may assume, at least computationally, that a cumulative distribution function (probability function,)  $P(x)$ ,

$$P(x) = \int_a^x p(y) dy, \quad (2.84)$$

can be inverted. That is, that  $P(x)$  is a single valued function in  $x$ , and we may meaningfully calculate with  $x(P)$ .

#### 2.9.4 Exponential Processes

To this end, we now want to consider a random process that is discrete in sequence. By this, we mean that we usually have a single thing and we measure its state repeatedly. As a concrete example, we consider how many times we should have to flip a coin to observe a head, or to roll a six-sided dice to observe a "5". We designate the counting of the trials, either coin flips or dice throws, in concrete terms by  $n, n = 1..$

Let the probability of the desired event be  $q$ , which for the coin should have the specific value  $1/2$  and for the dice  $1/6$ . The probability that we first observe the state on the  $n^{th}$  trial (observation) is  $P_n$ . For the first trial we have simply

$$P_1 = q. \quad (2.85)$$

Since we cannot observe the state on the first trial for it to be first observed on the second trial, the probability of observing the state on the second trial is just

$$P_2 = (1 - q)q, \quad (2.86)$$

which is just the product of the probability that the state does not occur on the first trial times the probability it does occur on the second trial.

From this we may deduce that the general form of the probability of first observing the state on the  $n^{\text{th}}$  trial is just

$$P_n = (1 - q)^{n-1} q. \quad (2.87)$$

We may now perform all manner of calculations since this amounts to the formulation of the discrete density function.

Before proceeding, it is useful to introduce a basic result of the infinitesimal calculus in the form of a quantity known as a **geometric series**,

$$\sum_{n=0}^N x^n = \frac{1 - x^{N+1}}{1 - x}, 0 \leq x < 1. \quad (2.88)$$

The equality  $x = 0$  follows since  $0^0 = 1!$  This series may be derived by expanding  $1/(1 - x)$  in a Maclaurin's series in  $x$ , but we readily accept the result here and such demonstration is left for adventurous students. Since we require  $x < 1$ , we may take the limit

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N x^n \rightarrow \frac{1}{1 - x}, \quad (2.89)$$

which is the more commonly used form.

If we now take the probabilities that the state is first observed on the  $n^{\text{th}}$  trial and sum over all possible trials, we have the series

$$\sum_{n=1}^{\infty} P_n = \sum_{n=1}^{\infty} (1 - q)^{n-1} q. \quad (2.90)$$

If we now rearrange the right hand side and change to a new index  $m = n - 1$ , we may rewrite this as

$$\sum_{n=1}^{\infty} P_n = q \sum_{m=0}^{\infty} (1 - q)^m. \quad (2.91)$$

We recognize this new right hand side as containing an infinite geometric series with  $x = 1 - q$  from which we may rewrite this equation as

$$\begin{aligned} \sum_{n=1}^{\infty} P_n &= \frac{q}{1 - (1 - q)}, \\ &= \frac{q}{q}, \\ &= 1, \end{aligned} \quad (2.92)$$

and we see the conservation of probability in action.

Now, we may ask the question of how we calculate the expected number of trials before we observe the state. This may be written simply as

$$\langle n \rangle = \sum_{n=1}^{\infty} n P_n, \quad (2.93)$$

which we see is just the value of the number of the observation times its probability, summed over all possible observations. This may immediately be rewritten as

$$\langle n \rangle = \sum_{n=1}^{\infty} n (1 - q)^{n-1} q, \quad (2.94)$$

which we could sum as is but that would be tedious and difficult. Instead, we shall make use of mathematical trickery.

By the property of the infinitesimal derivative, we may write

$$n (1 - q)^{n-1} = -\frac{d}{dq} (1 - q)^n, \quad (2.95)$$

which allows us to rewrite the sum as

$$\langle n \rangle = -q \frac{d}{dq} \sum_{n=1}^{\infty} (1 - q)^n, \quad (2.96)$$

where we have rearranged the sum and interchanged the order of derivative and sum. We can do the latter because the sum is essentially independent of the variable  $q$  so long as we do not violate the restrictions on the sum.

We now note that the sum differs from an infinite geometric series only by the  $n = 0$  term, so we may rewrite the series as

$$\langle n \rangle = -q \frac{d}{dq} \left[ \sum_{n=0}^{\infty} (1 - q)^n - 1 \right], \quad (2.97)$$

and we now have a geometric series. This is easily summed by inspection, giving us

$$\langle n \rangle = -q \frac{d}{dq} \left[ \frac{1}{q} - 1 \right], \quad (2.98)$$

from our previous calculation. Now we may differentiate the square bracket term by term and multiply by  $q$ , giving us

$$\langle n \rangle = \frac{1}{q}. \quad (2.99)$$

So the expected number of trials to observe a state of probability  $q$  is  $1/q$ .

A simple experiment may be conducted by tossing a coin until heads occurs. Each trail consists of noting the number of tosses necessary. Designate these observations as  $\{n_i, i = 1..I\}$ . As  $I$  gets larger, we find that

$$\lim_{I \rightarrow \infty} \frac{1}{I} \sum_{i=1}^I n_i \rightarrow \frac{1}{q}, \quad (2.100)$$

which should have a specific value of 2 for the coin toss. This is left as an exercise.

Now let us shift our view of this process. Consider now that each trial takes a set amount of time  $\tau$ . As a result, we may convert this from a discrete to a continuous process so long as  $q$  is sufficiently small.

Start with the probability of observation

$$P_n = (1 - q)^{n-1} q$$

and calculate the finite difference

$$\begin{aligned} \Delta P_n &= (1 - q)^n q - (1 - q)^{n-1} q & (2.101) \\ &= [(1 - q) - 1] (1 - q)^{n-1} q \\ &= [(1 - q) - 1] P_n \\ &= -q P_n. \end{aligned}$$

From this we may write the finite differential as

$$\frac{\Delta P_n}{\Delta n} = -q P_n, \quad (2.102)$$

since  $\Delta n = 1$ .

We now write the number of trials as

$$n = \frac{t}{\tau}, \quad (2.103)$$

to within a quantity less than one, so we may write the differential as

$$\Delta n = \frac{\Delta t}{\tau}. \quad (2.104)$$

This allows us to rewrite, by substitution, equation 2.103 as

$$\frac{\Delta P_t}{\Delta t} = -\frac{q}{\tau} P_t, \quad (2.105)$$

where we have made the somewhat unspecified transformation  $P_n \rightarrow P_t$ . This will clarify shortly.

If we now take the limit as  $\Delta t \rightarrow 0$ , we transform the finite differential into an infinitesimal differential, or derivative, and the discrete probability now become a continuous probability function with a differential equation

$$\frac{dP(t)}{dt} = -\frac{q}{\tau} P(t), \quad (2.106)$$

where  $P(t)$  now represents the probability (cdf) that by time  $t$  we have observed the state. If we now solve the differential equation,

$$P(t) = 1 - e^{-\frac{qt}{\tau}} \quad (2.107)$$

which we may differentiate to obtain the pdf of the process, which has form

$$p(t) = \frac{q}{\tau} e^{-\frac{qt}{\tau}}. \quad (2.108)$$

From this we may now compute the expected time to observation of the state as

$$\langle t \rangle = \frac{q}{\tau} \int_0^{\infty} t e^{-\frac{qt}{\tau}} dt, \quad (2.109)$$

which we may calculate as

$$\langle t \rangle = \frac{\tau}{q}. \quad (2.110)$$

This result, to within the difference between an integer and a real, is identical to the discrete result.

We have seen here that a discrete random process giving rise to a geometric distribution is in a sense equivalent to a continuous random process whose distribution is negative exponential. Such processes have some importance in physics and we shall explore them more closely in later chapters.

### 2.9.5 The Gaussian Distribution

We now consider the **Gaussian** (named after the noted mathematician Carl Friedrich Gauss 1777-1855) or **normal** distribution, so beloved of teachers and students. The pdf is

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right], \quad (2.111)$$

where the meanings of  $\bar{x}$  and  $\sigma$  will shortly be made clear. The cdf, as such, does not exist in closed form, but is expressed as the integral

$$P(y) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^y \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right] dx. \quad (2.112)$$

Because there is no closed form, before computers it was traditionally necessary to have to make use of tabulated computations.

The moments of the distribution may be calculated with the following integral taken from an integral table [Gradshteyn Ryzhik 1965, 3.462 2, p. 337]

$$\int_{-\infty}^{\infty} x^n \exp [-px^2 + 2qx] dx = \frac{1}{2^{n-1}p} \sqrt{\frac{\pi}{p}} \frac{d^{n-1}}{dq^{n-1}} \left( q \exp \left( \frac{q^2}{p} \right) \right). \quad (2.113)$$

We may calculate the mean of the Gaussian distribution as

$$\begin{aligned}
 \langle x \rangle &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^y x \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right] dx & (2.114) \\
 &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^y x \exp \left[ -\frac{x^2}{2\sigma^2} + \frac{2x\bar{x}}{2\sigma^2} - \frac{\bar{x}^2}{2\sigma^2} \right] dx \\
 &= \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{\bar{x}^2}{2\sigma^2} \right] \int_{-\infty}^y x \exp \left[ -\frac{x^2}{2\sigma^2} + \frac{2x\bar{x}}{2\sigma^2} \right] dx.
 \end{aligned}$$

At this point we recognize the integral as having the form of our borrowed integral, equation 2.113 with

$$\begin{aligned}
 n &= 1, \\
 p &= \frac{1}{2\sigma^2}, \\
 q &= \frac{\bar{x}}{2\sigma^2}.
 \end{aligned}$$

Hence, we have by substitution

$$\begin{aligned}
 \langle x \rangle &= \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{\bar{x}^2}{2\sigma^2} \right] \frac{1}{2^{1-1}p} \sqrt{\frac{\pi}{p}} \frac{d^{1-1}}{dq^{1-1}} \left( q \exp \left( \frac{q^2}{p} \right) \right) \Big|_{q,p}, & (2.115) \\
 &= \frac{1}{\sqrt{2\pi}\sigma} \exp \left[ -\frac{\bar{x}^2}{2\sigma^2} \right] 2\sigma^2 \sqrt{2\sigma^2\pi} \left( \frac{\bar{x}}{2\sigma^2} \exp \left( \frac{\bar{x}^2}{4\sigma^4} 2\sigma^2 \right) \right) \\
 &= \bar{x}.
 \end{aligned}$$

This shows us that  $\bar{x}$  is the mean and that the pdf is symmetric about the mean, unlike the exponential distribution.

The second moment may be calculated by exactly the same technique, giving us

$$\langle x^2 \rangle = \bar{x}^2 + \sigma^2, \quad (2.116)$$

by which we see that  $\sigma^2$  is the variance, as we suspected from the notation.

A word of warning is in order about the Gaussian distribution. Technically the random variable  $x$  has range  $[-\infty, \infty]$ . This means that  $x$  should be capable of taking on all values from negative to positive infinity. Sometimes we make approximations on different ranges such as when  $0 < \bar{x} \ll \sigma$ . Additionally, while the cdf does not exist in closed form, a quite good approximation exists that is useful for computational purposes. This is

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^y \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right] dx \simeq \frac{1}{2} \left[ 1 + \text{sign}(y - \bar{x}) \sqrt{1 - \exp \left[ -\frac{2(y - \bar{x})^2}{\pi\sigma^2} \right]} \right], \quad (2.117)$$

which is sometimes known as Feynman's approximation because of his great fondness for it.

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